

GLOBAL DIMENSION OF DIFFERENTIAL OPERATOR RINGS. II

BY

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ABSTRACT. The aim of this paper is to find the global homological dimension of the ring of linear differential operators $R[\theta_1, \dots, \theta_u]$ over a differential ring R with u commuting derivations. When R is a commutative noetherian ring with finite global dimension, the main theorem of this paper (Theorem 21) shows that the global dimension of $R[\theta_1, \dots, \theta_u]$ is the maximum of k and $q + u$, where q is the supremum of the ranks of all maximal ideals M of R for which R/M has positive characteristic, and k is the supremum of the sums $\text{rank}(P) + \text{diff dim}(P)$ for all prime ideals P of R such that R/P has characteristic zero. [The value $\text{diff dim}(P)$ is an invariant measuring the differentiability of P in a manner defined in §3.] In case we are considering only a single derivation on R , this theorem leads to the result that the global dimension of $R[\theta]$ is the supremum of $\text{gl dim}(R)$ together with one plus the projective dimensions of the modules R/J , where J is any primary differential ideal of R . One application of these results derives the global dimension of the Weyl algebra in any degree over any commutative noetherian ring with finite global dimension.

1. Introduction. As in [5], we reserve the term *differential ring* for a nonzero associative ring R with unit together with a single specified derivation δ on R . In case we have specified a finite collection $\delta_1, \dots, \delta_u$ of commuting derivations on R , we shall refer to R as a *u -differential ring*. The *ring of differential operators* over a u -differential ring R is additively the group of all polynomials over R in indeterminates $\theta_1, \dots, \theta_u$, with multiplication subject to the requirements $\theta_i \theta_j = \theta_j \theta_i$ for all i, j , and $\theta_i a = a \theta_i + \delta_i a$ for all i , all $a \in R$. We denote this ring by $R[\theta_1, \dots, \theta_u]$, or by $R[\theta]$ in the case of a single derivation. The elements of $R[\theta_1, \dots, \theta_u]$ are normally written as sums of monomials of the form rp , where $r \in R$ and p is a product of powers of the θ_i , although for some arguments it is more convenient to use right-hand coefficients. (Note that when an element of $R[\theta_1, \dots, \theta_u]$ is written with left-hand coefficients, these coefficients will in general be different from those used to express

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the element with right-hand coefficients.) In particular, any element $x \in R[\theta]$ is written as $x = r_0 + r_1\theta + \dots + r_n\theta^n$ for suitable $r_i \in R$, and when $r_n \neq 0$ we say that n is the *degree* of x and that r_n is the *leading coefficient* of x . Finally, for induction purposes we note that

$$R[\theta_1, \dots, \theta_u] = R[\theta_1, \dots, \theta_{u-1}][\theta_u],$$

where δ_u has been implicitly extended to $R[\theta_1, \dots, \theta_{u-1}]$ by setting $\delta_u\theta_i = 0$ for all i .

The objective of this paper is to derive formulas for the global dimension of $R[\theta_1, \dots, \theta_u]$, where R is a commutative noetherian u -differential ring with finite global dimension. Basically, the task breaks down into the problems of finding suitable lower bounds and upper bounds for the global dimension of $R[\theta_1, \dots, \theta_u]$. Since these two problems require relatively different techniques, we allot separate sections of the paper to them. In both cases we also require the techniques of localization: namely ordinary localization of the commutative ring R at a prime ideal, which induces a natural noncommutative localization on the ring $R[\theta_1, \dots, \theta_u]$.

Our notation for the various homological dimensions involved with a ring S is as follows: $\text{r gl dim } S$ denotes the right global dimension of S , and $\text{GWD}(S)$ denotes the global weak dimension of S . For any S -module A , we use $\text{pd}_S(A)$ and $\text{wd}_S(A)$ to stand for the respective projective and weak dimensions of A . The reason that weak dimensions are useful is that we shall be dealing mostly with noetherian rings. For if R is a right and left noetherian differential ring, then $R[\theta]$ is right and left noetherian, as observed in [2, p. 68]. By induction, $R[\theta_1, \dots, \theta_u]$ is right and left noetherian also. Our basic estimates on homological dimensions are given in the following two propositions, which follow automatically by induction from [5, Propositions 2, 3].

PROPOSITION 1. *Let R be any u -differential ring, and set $T = R[\theta_1, \dots, \theta_u]$. If A is any right T -module, then*

$$\text{pd}_R(A) \leq \text{pd}_T(A) \leq u + \text{pd}_R(A).$$

PROPOSITION 2. *If R is any u -differential ring with $\text{r gl dim } R < \infty$, then*

$$\text{r gl dim } R \leq \text{r gl dim } R[\theta_1, \dots, \theta_u] \leq u + \text{r gl dim } R.$$

The left-hand inequality in Proposition 2 may fail if $\text{r gl dim } R = \infty$, as shown in [5, §2].

We close this section with two propositions which give the basic results on the localization procedures needed later. The first of these is proved in exactly the same manner as [5, Lemma 7].

PROPOSITION 3. *Let R be any commutative u -differential ring, and set $T = R[\theta_1, \dots, \theta_u]$. If S is any multiplicatively closed subset of R , then the following are true:*

- (a) *Each δ_i induces a derivation on R_S according to the rule $\delta_i(r/s) = [(\delta_i r)s - r(\delta_i s)]/s^2$.*
- (b) *The natural map $T \rightarrow R_S[\theta_1, \dots, \theta_u]$ makes $R_S[\theta_1, \dots, \theta_u]$ into a flat right and left T -module such that the multiplication map $R_S[\theta_1, \dots, \theta_u] \otimes_T R_S[\theta_1, \dots, \theta_u] \rightarrow R_S[\theta_1, \dots, \theta_u]$ is an isomorphism.*
- (c) $\text{r gl dim } R_S[\theta_1, \dots, \theta_u] \leq \text{r gl dim } T$.

PROPOSITION 4. *Let R be a commutative noetherian u -differential ring with $\text{gl dim } R < \infty$. Then*

$$\begin{aligned} \text{r gl dim } R[\theta_1, \dots, \theta_u] \\ = \sup\{\text{r gl dim } R_M[\theta_1, \dots, \theta_u] \mid M \text{ is a maximal ideal of } R\}. \end{aligned}$$

PROOF. Inasmuch as all rings involved in this proposition are right noetherian, it suffices to prove the corresponding statement for global weak dimension. Just as in the proof of [5, Lemma 7], we see that each of the rings $R_M[\theta_1, \dots, \theta_u]$ is a classical localization of $R[\theta_1, \dots, \theta_u]$ with respect to the multiplicative set $R \setminus M$. It is easily checked that these localizations satisfy the hypotheses of [13, Proposition 1], from which we obtain the desired result.

2. Lower bounds. In this section we set up our basic tool for finding lower bounds for the global dimension of $R[\theta_1, \dots, \theta_u]$. This is Theorem 7, which allows us to compute the projective dimensions of those $R[\theta_1, \dots, \theta_u]$ -modules which happen to be finitely generated as R -modules. As one consequence, we find that $\text{r gl dim } R[\theta_1, \dots, \theta_u] \geq u + \text{rank}(M)$ for any maximal ideal M of R such that R/M has positive characteristic. We begin with two lemmas, the first of which is essentially a special case of [6, Lemma, p. 68].

LEMMA 5. *Let R be any differential ring, and let A be a right $R[\theta]$ -module. If $E: 0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ is an exact sequence of right R -modules with F_R free, then K and F can be made into right $R[\theta]$ -modules such that E becomes an exact sequence of $R[\theta]$ -modules.*

PROOF. Let $f: K \rightarrow F$ and $g: F \rightarrow A$ denote the maps in E . Choosing a decomposition of F as a direct sum of copies of R , and applying δ to each copy of R , we obtain an additive map $d: F \rightarrow F$ such that $d(xr) = (dx)r + x(\delta r)$ for all $x \in F$, $r \in R$. Define a map $h: F \rightarrow A$ by the rule $hx = g(dx) + (gx)\theta$, and check that h is an R -homomorphism. Then h lifts to an R -homomorphism $k: F \rightarrow F$ such that $gk = h$.

Now $d' = k - d$ is an additive endomorphism of F such that $d'(xr) = (d'x)r - x(\delta r)$ for all $x \in F$, $r \in R$, from which we infer that F can be made into a right $R[\theta]$ -module by defining $x\theta = d'x$ for all $x \in F$. Computing that now $g(x\theta) = (gx)\theta$ for all $x \in F$, we see that g is an $R[\theta]$ -homomorphism. As a consequence, $\ker g$ is an $R[\theta]$ -submodule of F , hence K can be made into a right $R[\theta]$ -module so that f is an $R[\theta]$ -homomorphism.

LEMMA 6. *Let R be a semiprime left Goldie differential ring. If J is any essential left ideal of $R[\theta]$, then J contains an element of $R[\theta]$ whose leading coefficient is a regular element of R .*

PROOF. Since R is left Goldie, it must contain a finite direct sum $A_1 \oplus \dots \oplus A_k$ of nonzero uniform left ideals which is essential in ${}_R R$. The essentiality of J implies that each of the left ideals $R[\theta]A_i$ must contain a nonzero element x_i from J . After multiplying the x_i on the left by suitable powers of θ , we may assume that the x_i all have the same degree, say n . Inasmuch as $R[\theta] = R + \theta R + \theta^2 R + \dots$, we see that $R[\theta]A_i = A_i + \theta A_i + \theta^2 A_i + \dots$. Noting that the degree of x_i remains the same when x_i is written with coefficients on the right, we see that $x_i = x_{i0} + \theta x_{i1} + \dots + \theta^{n-1} x_{i,n-1} + \theta^n a_i$ for some x_{ij} , $a_i \in A_i$, $a_i \neq 0$. Changing back to left-hand coefficients, the leading coefficient of x_i is still a_i , although the other coefficients need not even belong to A_i .

Now Ra_i is a nonzero submodule of the uniform left ideal A_i and hence is essential in A_i , from which we deduce that $Ra_1 \oplus \dots \oplus Ra_k$ is an essential left ideal of R . Inasmuch as R is a semiprime left Goldie ring, [8, Lemma 7.2.5] says that $Ra_1 \oplus \dots \oplus Ra_k$ must contain a regular element a of R , say $a = r_1 a_1 + \dots + r_k a_k$. Since each x_i has leading term $a_i \theta^n$, we now conclude that $r_1 x_1 + \dots + r_k x_k$ is an element of J whose leading coefficient is a .

THEOREM 7. *Let R be a semiprime right and left noetherian u -differential ring, and set $T = R[\theta_1, \dots, \theta_u]$. If A is any nonzero right T -module such that A_R is finitely generated, then $\text{pd}_T(A) = u + \text{pd}_R(A)$.*

PROOF. Each of the rings $T_j = R[\theta_1, \dots, \theta_j]$ is right and left noetherian, and it is easily checked that each T_j is semiprime as well. Now A is a finitely generated right T_j -module for each j , and we are done if we show that the projective dimension of A over each T_{j+1} is exactly one greater than the projective dimension of A over T_j . Thus it suffices to consider only the 1-differential case: here R is a semiprime right and left noetherian differential ring, A is a nonzero right $R[\theta]$ -module such that A_R is finitely generated, and we must prove that $\text{pd}_{R[\theta]}(A) = 1 + \text{pd}_R(A)$.

The case $\text{pd}_R(A) = \infty$ is taken care of by Proposition 1, hence we may assume that $\text{pd}_R(A) = n < \infty$, and we induct on n . As noted above, $R[\theta]$ is a semiprime right and left noetherian ring, hence the maximal right quotient ring Q of $R[\theta]$ coincides with the maximal left quotient ring of $R[\theta]$ (and is a classical right and left quotient ring). Also, $R[\theta]$ is a semiprime right Goldie ring, hence [14, Theorem 1.7] shows that $R[\theta]$ is a right nonsingular ring.

If $n = 0$, then $\text{pd}_{R[\theta]}(A) \leq 1$ by Proposition 1; hence it remains to show that $A_{R[\theta]}$ is not projective. Inasmuch as $A \neq 0$ and all projective right $R[\theta]$ -modules are nonsingular, it suffices to show that $A_{R[\theta]}$ is singular. Given any $a \in A$, set $J = \{x \in R[\theta] \mid ax = 0\}$ and note that $R[\theta]/J$ is noetherian as an R -module. Now any nonzero right ideal K of $R[\theta]$ contains elements of arbitrarily high degree, whence K_R cannot be finitely generated. Thus the natural map $K \rightarrow R[\theta] \rightarrow R[\theta]/J$ cannot be a monomorphism, i.e., $K \cap J \neq 0$. Therefore J is an essential right ideal of $R[\theta]$ and so A is indeed a singular $R[\theta]$ -module.

Next assume that $n = 1$, and choose a positive integer k such that A_R can be generated by k elements. If S denotes the ring of all $k \times k$ matrices over R , then we obtain a Morita equivalence between the category of all right R -modules and the category of all right S -modules, where any right R -module B gets taken to $B \otimes_R R^k$, i.e., to B^k . We intend to use this equivalence to transfer our problem to S -modules, since A^k is a cyclic right S -module. Now δ can be extended to a derivation of S by letting δ act on each entry of any matrix in S , and then $S[\theta]$ may be identified with the ring of all $k \times k$ matrices over $R[\theta]$. With this identification, we get another Morita equivalence between the category of all right $R[\theta]$ -modules and the category of all right $S[\theta]$ -modules, where any right $R[\theta]$ -module B gets taken to B^k . Because of these equivalences, $\text{pd}_S(A^k) = 1$ and $\text{pd}_{R[\theta]}(A) = \text{pd}_{S[\theta]}(A^k)$, hence we may assume without loss of generality that A_R is cyclic.

Therefore we may assume that $A = R/I$ for some right ideal I of R . Inasmuch as A is also a right $R[\theta]$ -module, we have $\bar{1}\theta = \bar{\alpha}$ for some $\alpha \in R$. Then $\bar{r}\theta = \overline{(\alpha - \delta)r}$ for all $r \in R$ and consequently $(\alpha - \delta)I \subseteq I$. Noting that $R[\theta] = R + (\theta - \alpha)R[\theta]$, we see that $A \cong R[\theta]/J$, where $J = I + (\theta - \alpha)R[\theta]$.

We claim that for any $R[\theta]$ -homomorphism $f: J \rightarrow R[\theta]$, $f|_I$ must be left multiplication by some element of $R[\theta]$. Since $R[\theta]$ is a right nonsingular ring, its maximal right quotient ring Q is the injective hull of $R[\theta]_{R[\theta]}$, hence f must be left multiplication by some $t \in Q$. Noting that $t(\theta - \alpha) \in R[\theta]$, we see that $t = x(\theta - \alpha)^{-1}$ for some $x \in R[\theta]$. This element x can be put in the form $x = x_0 + x_1(\theta - \alpha)$ for suitable $x_0 \in R$ and $x_1 \in R[\theta]$, whence $t = x_0(\theta - \alpha)^{-1} + x_1$. If $x_0 = 0$, then f itself is left multiplication by the

element $x_1 \in R[\theta]$ and the claim holds, hence we may assume that $x_0 \neq 0$. We have $tJ = fJ \subseteq R[\theta]$, and clearly $x_1J \subseteq R[\theta]$ as well, whence $x_0(\theta - \alpha)^{-1}J \subseteq R[\theta]$.

Inasmuch as Q is also the maximal left quotient ring of $R[\theta]$, we must have $Kx_0(\theta - \alpha)^{-1} \subseteq R[\theta]$ for some essential left ideal K of $R[\theta]$, and by Lemma 6, K must contain an element y whose leading coefficient is a regular element of R . Now y is clearly a regular element of $R[\theta]$ and so is invertible in Q , hence we obtain $x_0(\theta - \alpha)^{-1} = y^{-1}z$ for some $z \in R[\theta]$, or $yx_0 = z(\theta - \alpha)$. Since $x_0 \neq 0$ we have $z \neq 0$, too, which makes it possible to talk about the degrees of the elements in this last equation. Obviously $\deg[z(\theta - \alpha)] = 1 + \deg(z)$, and since the leading coefficient of y is a regular element we obtain $\deg(yx_0) = \deg(y)$; thus $\deg(y) = 1 + \deg(z)$. Given any $r \in I$, we have $y^{-1}zr = x_0(\theta - \alpha)^{-1}r \in R[\theta]$ (because $r \in J$), whence $zr \in yR[\theta]$. Since $\deg(y) > \deg(z)$, and since $\deg(yw) \geq \deg(y)$ for all nonzero $w \in R[\theta]$, this is possible only when $zr = 0$. Thus we obtain $zI = 0$, from which we infer that $x_0(\theta - \alpha)^{-1}I = 0$. It follows that $f|_I$ is just left multiplication by the element $x_1 \in R[\theta]$, as claimed.

As right R -modules, $J = I \oplus (\theta - \alpha)R[\theta]$, from which we see that I can be made into a right $R[\theta]$ -module so that the projection $p: J \rightarrow I$ is an $R[\theta]$ -homomorphism. Choose an $R[\theta]$ -epimorphism $g: F \rightarrow I$, where F is a finitely generated free right $R[\theta]$ -module. If we assume that $J_{R[\theta]}$ is projective, then p must lift to an $R[\theta]$ -homomorphism $h: J \rightarrow F$ such that $gh = p$. In view of the claim just proved above, we see that $h|_I$ must be left multiplication by some $w \in F$, from which we compute that $(gw)r = r$ for all $r \in I$. Consequently gw is an idempotent and $(gw)R = I$, hence $(R/I)_R$ must be projective. However, this contradicts the assumption that $\text{pd}_R(A) = 1$, and thus $J_{R[\theta]}$ cannot be projective. This gives us $\text{pd}_{R[\theta]}(A) > 1$, so by Proposition 1 we conclude that $\text{pd}_{R[\theta]}(A) = 2$.

Finally, let $n > 1$ and assume the theorem holds for $n - 1$. Choose an exact sequence $E: 0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ of right R -modules with F_R finitely generated free, and use Lemma 5 to make E into an exact sequence of right $R[\theta]$ -modules. Now K is a right $R[\theta]$ -module which is finitely generated as an R -module, and $\text{pd}_R(K) = n - 1 > 0$ (so that in particular $K \neq 0$), hence we obtain $\text{pd}_{R[\theta]}(K) = n$ from the induction hypothesis. Inasmuch as $n > 1$ and $\text{pd}_{R[\theta]}(F) \leq 1$ by Proposition 1, it now follows from the long exact sequence for Ext that $\text{pd}_{R[\theta]}(A) = n + 1$. $\square$

Using more homological methods, a stronger version of Theorem 7 has been proved in [12, Corollary 1.7(b)].

I am grateful to the referee for pointing out the necessity of condition (b) in the following corollary.

COROLLARY 8. *Let R be a semiprime right and left noetherian u -differential ring, let J be a proper right ideal of R , and let $\alpha_1, \dots, \alpha_u \in R$ such that*

$$(a) \quad (\delta_i - \alpha_i)(J) \subseteq J \text{ for } i = 1, \dots, u,$$

$$(b) \quad (\delta_i - \alpha_i)(\alpha_j) - (\delta_j - \alpha_j)(\alpha_i) \in J \text{ for } i, j = 1, \dots, u.$$

Then $\text{rgl dim } R[\theta_1, \dots, \theta_u] \geq u + \text{pd}_R(R/J)$.

PROOF. Let $A = R/J$, which is a nonzero finitely generated right R -module. Using (a) and (b), we infer that A can be made into a right $R[\theta_1, \dots, \theta_u]$ -module by setting $\overline{r}\theta_i = \overline{(\alpha_i - \delta_i)r}$ for all i and all $r \in R$. (Condition (a) ensures that $x\theta_i$ is well defined, and condition (b) ensures that $x\theta_i\theta_j = x\theta_j\theta_i$. The details are very straightforward.) Consequently, Theorem 7 says that A is a right $R[\theta_1, \dots, \theta_u]$ -module with projective dimension $u + \text{pd}_R(A)$. $\square$

Corollary 8 applies in particular to the case when J is a differential right ideal of R , i.e., $\delta_i(J) \subseteq J$ for all i . In this case, condition (b) is trivially satisfied.

In order to apply Theorem 7 or Corollary 8 in the case when R is a commutative noetherian ring of finite global dimension, we must know that R is semiprime. This is probably well known, as are the other facts in the following proposition, which we include for completeness.

PROPOSITION 9. *Let R be any commutative noetherian ring with $\text{gl dim } R = n < \infty$.*

(a) *R is a finite direct product of integral domains, and thus is a semiprime ring.*

(b) *If M is any maximal ideal of R , then $\text{gl dim } R_M = \text{rank}(M) = \text{pd}_R(R/M) \leq n$.*

(c) *The (classical) Krull dimension of R is n .*

PROOF. (a) For each maximal ideal M of R [9, Part III, Theorem 11] says that $\text{gl dim } R_M \leq n < \infty$, hence it follows from [9, Part III, Theorem 13] that R_M is a regular local ring. Thus R_M is an integral domain for every maximal ideal M [10, Theorem 164], whence [10, Theorem 168] says that R is a finite direct product of integral domains.

(b) As seen in (a), R_M is a regular local ring. According to [9, Part III, Theorem 12], $\text{gl dim } R_M$ is the same as the Krull dimension of R_M , i.e., $\text{gl dim } R_M = \text{rank}(M)$. In view of [10, Theorem 176], we also see that the projective dimension of R_M/MR_M over R_M is equal to $\text{rank}(M)$. Inasmuch as

R is noetherian, the projective dimension of any finitely generated R -module A is the supremum of the projective dimensions of the R_K -modules A_K , where K ranges over all maximal ideals of R . For the case $A = R/M$, we have $A_M = R_M/MR_M$ and $A_K = 0$ for all other K , from which we conclude that $\text{pd}_R(R/M) = \text{rank}(M)$.

(c) Since R is noetherian, n is the supremum of the numbers $\text{gl dim } R_M$ over all maximal ideals M , hence (c) follows immediately from (b).

We conclude this section by deriving the lower bound $u + \text{rank}(M) \leq \text{r gl dim } R[\theta_1, \dots, \theta_u]$, where M is any maximal ideal of R such that R/M has positive characteristic. We must also derive lower bounds for $\text{r gl dim } R[\theta_1, \dots, \theta_u]$ related to maximal ideals M such that R/M has characteristic zero, but this depends on the differential dimension of M , which we develop in the next section.

PROPOSITION 10. *Let R be a commutative noetherian u -differential ring with $\text{gl dim } R < \infty$, and let M be a maximal ideal of R . If R/M has characteristic $p > 0$, then*

$$\text{r gl dim } R[\theta_1, \dots, \theta_u] \geq u + \text{rank}(M).$$

PROOF. According to Proposition 9, the simple module R/M satisfies the property $\text{pd}_R(R/M) = \text{rank}(M) < \infty$. If A is any nonzero R -module with a composition series such that all the composition factors are isomorphic to R/M , then it follows from the long exact sequence for Ext (by induction on length) that $\text{pd}_R(A) = \text{rank}(M)$.

Now let J be the ideal of R generated by pR and $\{x^p \mid x \in M\}$, and note that $\delta_i(J) \subseteq J$ for all $i = 1, \dots, u$. Since $\text{char}(R/M) = p$, we see that $J \subseteq M$, whence $R/J \neq 0$. Inasmuch as M/J is a nil ideal in the noetherian ring R/J , Levitzki's Theorem says that M/J must be nilpotent, from which we infer that R/J has a composition series with all composition factors isomorphic to R/M . Now $\text{pd}_R(R/J) = \text{rank}(M)$, hence the desired inequality follows from Corollary 8.

COROLLARY 11. *Let R be a commutative noetherian u -differential ring with $\text{gl dim } R = n < \infty$. If R has positive characteristic, then*

$$\text{r gl dim } R[\theta_1, \dots, \theta_u] = n + u.$$

PROOF. In view of Proposition 9, we must have $\text{rank}(M) = n$ for some maximal ideal M , whence Proposition 10 yields $\text{r gl dim } R[\theta_1, \dots, \theta_u] \geq n + u$. According to Proposition 1, we also have $\text{r gl dim } R[\theta_1, \dots, \theta_u] \leq n + u$.

3. Differential dimension. The purpose of this section is to introduce a concept of differential dimension for prime ideals P of R , and to obtain the lower bounds

$$\text{rank}(P) + \text{diff dim}(P) \leq \text{rgl dim } R[\theta_1, \dots, \theta_u].$$

This differential dimension of P is meant to measure the “differentiability” of P in the sense that it indicates how large a collection of R -linear combinations of the derivations $\delta_1, \dots, \delta_u$ can map P into itself. In particular, the differential dimension of P will be u if and only if P is closed under all the δ_i . The details follow.

Given any commutative u -differential ring R , make $\text{Hom}_Z(R, R)$ into a left R -module by defining $(rf)(x) = r(fx)$ for all $r, x \in R, f \in \text{Hom}_Z(R, R)$, and let Δ denote the left R -submodule of $\text{Hom}_Z(R, R)$ generated by $\delta_1, \dots, \delta_u$. For any prime ideal P of R , the set $D(P) = \{f \in \Delta \mid f(P) \subseteq P\}$ is a left R -submodule of Δ , and it is clear that $\Delta/D(P)$ is a torsion-free left (R/P) -module. We define the *differential codimension* of P , abbreviated $\text{diff codim}(P)$, to be the rank of this torsion-free (R/P) -module $\Delta/D(P)$, i.e., the vector space dimension $[\mathcal{Q}[\Delta/D(P)]: \mathcal{Q}]$, where $\mathcal{Q}$ stands for the quotient field of R/P . [Alternately, $\text{diff codim}(P)$ may be defined as the Goldie dimension of the left R -module $\Delta/D(P)$.] Finally, we define the *differential dimension* of P , denoted $\text{diff dim}(P)$, to be $u - \text{diff codim}(P)$.

PROPOSITION 12. *Let R be a commutative u -differential ring. Let P be any prime ideal of R , and set $S = R_P, M = PR_P$. Then each δ_i induces a linear transformation δ_i^* in the dual space $V = \text{Hom}_{S/M}(M/M^2, S/M)$, and the subspace W of V spanned by $\delta_1^*, \dots, \delta_u^*$ has dimension exactly $\text{diff codim}(P)$.*

PROOF. Each δ_i induces a derivation on S as in Proposition 3, and this gives us additive maps $\delta_i: M \rightarrow S$. Observing that $\delta_i(M^2) \subseteq M$, we see that δ_i induces an additive map $\delta_i^*: M/M^2 \rightarrow S/M$, and an easy check confirms that δ_i^* is an (S/M) -homomorphism.

There is a left R -homomorphism $\phi: \Delta \rightarrow W$ such that $\phi(\delta_i) = \delta_i^*$ for each i , and an easy computation shows that $\ker \phi = D(P)$. Now $\phi\Delta$ is a left module over the domain $T = (R + M)/M \cong R/P$, from which we infer that ${}_T(\phi\Delta)$ and $_{R/P}[\Delta/D(P)]$ have the same rank, i.e., ${}_T(\phi\Delta)$ has rank $\text{diff codim}(P)$. Inasmuch as ${}_T(\phi\Delta)$ is torsion-free and S/M is the quotient field of T , the rank of ${}_T(\phi\Delta)$ is just $[S(\phi\Delta): S/M]$. Observing that $S(\phi\Delta) = W$, we conclude that $[W: S/M] = \text{diff codim}(P)$.

COROLLARY 13. *Let R be a commutative u -differential ring. If $P \subseteq Q$ are prime ideals of R , then $\text{diff codim}(PR_Q) = \text{diff codim}(P)$.*

PROOF. Inasmuch as the localization of R_Q at the prime ideal PR_Q is just R_P , this follows immediately from Proposition 12. $\square$

In particular, Corollary 13 shows that $\text{diff codim}(P) = \text{diff codim}(PR_P)$ for any prime ideal P , which makes it possible to carry out some computations using the maximal ideal PR_P in the local ring R_P . Before proving the inequality $\text{rank}(P) + \text{diff dim}(P) \leq \text{r gl dim } R[\theta_1, \dots, \theta_u]$, we introduce the following easy lemma, which will also be useful later.

LEMMA 14. (a) Let R be any ring such that $\text{r gl dim } R = n < \infty$. If $A \subseteq B$ are right R -modules with $\text{pd}_R(A) = n$, then $\text{pd}_R(B) = n$.

(b) Let R be any ring such that $\text{GWD}(R) = n < \infty$. If $A \subseteq B$ are R -modules with $\text{wd}_R(A) = n$, then $\text{wd}_R(B) = n$.

PROOF. (a) If $\text{pd}_R(B) < n$, then it follows from the long exact sequence for Ext that $\text{pd}_R(B/A) = n + 1$, which is impossible. (b) is proved similarly.

PROPOSITION 15. Let R be a commutative noetherian u -differential ring with $\text{gl dim } R < \infty$. If P is any prime ideal of R , then

$$\text{r gl dim } R[\theta_1, \dots, \theta_u] \geq \text{rank}(P) + \text{diff dim}(P).$$

PROOF. The local ring R_P is a commutative noetherian u -differential ring with $\text{gl dim } R_P < \infty$ and certainly $\text{rank}(PR_P) = \text{rank}(P)$. Inasmuch as $\text{diff dim}(PR_P) = \text{diff dim}(P)$ by Corollary 13 and $\text{r gl dim } R_P[\theta_1, \dots, \theta_u] \leq \text{r gl dim } R[\theta_1, \dots, \theta_u]$ by Proposition 3, it suffices to consider the case when R is local and P is its maximal ideal. According to Proposition 9, we have $\text{gl dim } R = \text{rank}(P) = \text{pd}_R(R/P)$; let n denote this common value.

If $s = \text{diff codim}(P)$, then Proposition 12 shows that the subspace W of $\text{Hom}_{R/P}(P/P^2, R/P)$ spanned by the induced linear transformations $\delta_1^*, \dots, \delta_u^*$ has dimension s . Thus W must have a basis consisting of s of the δ_i^* , hence we may arrange the indices $1, \dots, u$ so that $\delta_1^*, \dots, \delta_s^*$ is a basis for W .

Since R is semiprime by Proposition 9, the ring $Q = R[\theta_1, \dots, \theta_s]$ must be a semiprime ring, as well as right and left noetherian, and of course $R[\theta_1, \dots, \theta_u] = Q[\theta_{s+1}, \dots, \theta_u]$. Now PQ is a right ideal of Q and $Q/PQ \cong (R/P) \otimes_R Q$, whence $\text{pd}_Q(Q/PQ) \leq \text{pd}_R(R/P) = n$. On the other hand, since Q/PQ contains an R -submodule isomorphic to R/P , we obtain $\text{pd}_R(Q/PQ) = n$ from Lemma 14, and then Proposition 1 says that $\text{pd}_Q(Q/PQ) \geq n$. Therefore $\text{pd}_Q(Q/PQ) = n$.

Given any $j \in \{s+1, \dots, u\}$, we must have $\delta_j^* = r_{j1}\delta_1^* + \dots + r_{js}\delta_s^*$ for suitable $r_{ji} \in R$, whence $(\delta_j - r_{j1}\delta_1 - \dots - r_{js}\delta_s)(P) \subseteq P$. Setting

$q_j = r_{j1}\theta_1 + \dots + r_{js}\theta_s \in Q$, we compute that $(\delta_j - q_j)(PQ) \subseteq PQ$. Given any $i, j \in \{s+1, \dots, u\}$, we have

$$\left(\delta_i - \sum_{k=1}^s r_{ik}\delta_k\right)(P) \subseteq P \quad \text{and} \quad \left(\delta_j - \sum_{t=1}^s r_{jt}\delta_t\right)(P) \subseteq P,$$

from which it follows that

$$\begin{aligned} & \left[\left(\delta_j - \sum_{t=1}^s r_{jt}\delta_t\right) \left(\delta_i - \sum_{k=1}^s r_{ik}\delta_k\right) \right. \\ & \quad \left. - \left(\delta_i - \sum_{k=1}^s r_{ik}\delta_k\right) \left(\delta_j - \sum_{t=1}^s r_{jt}\delta_t\right) \right] (P) \subseteq P. \end{aligned}$$

We compute that

$$\begin{aligned} & \left(\delta_j - \sum_{t=1}^s r_{jt}\delta_t\right) \left(\delta_i - \sum_{k=1}^s r_{ik}\delta_k\right) - \left(\delta_i - \sum_{k=1}^s r_{ik}\delta_k\right) \left(\delta_j - \sum_{t=1}^s r_{jt}\delta_t\right) \\ &= \sum_{k=1}^s \left[\left(\delta_i - \sum_{t=1}^s r_{it}\delta_t\right)(r_{jk}) - \left(\delta_j - \sum_{t=1}^s r_{jt}\delta_t\right)(r_{ik}) \right] \delta_k; \end{aligned}$$

hence we obtain

$$\sum_{k=1}^s \left[\left(\delta_i - \sum_{t=1}^s r_{it}\delta_t\right)(r_{jk}) - \left(\delta_j - \sum_{t=1}^s r_{jt}\delta_t\right)(r_{ik}) \right] \delta_k^* = 0.$$

Inasmuch as $\delta_1^*, \dots, \delta_s^*$ are linearly independent over R/P , we see that

$$\left(\delta_i - \sum_{t=1}^s r_{it}\delta_t\right)(r_{jk}) - \left(\delta_j - \sum_{t=1}^s r_{jt}\delta_t\right)(r_{ik}) \in P \quad \text{for } k = 1, \dots, s,$$

from which we compute that $(\delta_i - q_i)(q_j) - (\delta_j - q_j)(q_i) \in PQ$. According to Corollary 8, we obtain $\text{r gl dim } Q[\theta_{s+1}, \dots, \theta_u] \geq u - s + n$. Inasmuch as $u - s = \text{diff dim}(P)$ and $n = \text{rank}(P)$, we are done.

4. Upper bounds. The purpose of this section is to introduce two kinds of upper bounds which are needed in the computation of the global dimension of $R[\theta_1, \dots, \theta_u]$. First, we prove a theorem which shows that the global dimension of $R[\theta_1, \dots, \theta_u]$ is the supremum of the projective dimensions of its simple modules. The second upper bound, which is needed only in the case

that R is an algebra over the rationals, shows that, for any maximal ideal M of R , all factor modules of $R[\theta_1, \dots, \theta_u]/MR[\theta_1, \dots, \theta_u]$ have projective dimension at most $\text{rank}(M) + \text{diff dim}(M)$.

For the first theorem, we need the concepts of Krull dimension (for non-commutative rings) and critical modules, as defined in [7].

THEOREM 16. *Let R be any nonzero right noetherian, left coherent ring. If $\text{r gl dim } R = n < \infty$, then $n = \sup\{\text{pd}_R(A) \mid A_R \text{ is simple}\}$.*

PROOF. Since this is clear for $n = 0$, we may assume that $n > 0$. Inasmuch as R is right noetherian, we have $\text{GWD}(R) = n$ and $\text{pd}_R(A) = \text{wd}_R(A)$ for all simple modules A_R , hence it suffices to show that R has a simple right module with weak dimension n . According to [3, Theorem 2.1], all direct products of flat right R -modules are flat, from which we infer that the weak dimension of any direct product of right R -modules equals the supremum of the weak dimensions of the factors.

In view of [7, Proposition 1.3], all finitely generated right R -modules have Krull dimension, and there certainly exist finitely generated right R -modules with weak dimension n . Now let α be minimal among the Krull dimensions of those finitely generated right R -modules which have weak dimension n , and choose some finitely generated right R -module B such that $\text{K dim}(B) = \alpha$ and $\text{wd}_R(B) = n$. Since $n > 0$, we have $B \neq 0$. All factor modules of B are finitely generated and hence have Krull dimension, whence [7, Theorem 2.1] says that every nonzero factor module of B contains a critical submodule. Thus B must have a chain of submodules $B_0 = 0 < B_1 < \dots < B_k = B$ such that each B_i/B_{i-1} is critical. Inasmuch as $\text{wd}_R(B) \leq \sup\{\text{wd}_R(B_i/B_{i-1})\}$, we must have $\text{wd}_R(B_i/B_{i-1}) = n$ for some i . Setting $A = B_i/B_{i-1}$, we see by [7, Lemma 1.1] that $\text{K dim}(A) \leq \alpha$, hence it follows from the minimality of α that $\text{K dim}(A) = \alpha$.

We now have a finitely generated α -critical right R -module A such that $\text{wd}_R(A) = n$. We claim that $\alpha = 0$, i.e., that A is simple.

Assume on the contrary that $\alpha > 0$. Then every nonzero submodule of A is α -critical too [7, Proposition 2.3], and thus is not simple; so A has no simple submodules. Thus the intersection of all nonzero submodules of A is zero, hence we obtain an embedding $A \rightarrow P$, where P is the direct product of all proper factors of A . Since A is α -critical, each proper factor of A is a finitely generated module with Krull dimension strictly less than α , so by the minimality of α we see that each proper factor of A has weak dimension at most $n - 1$. However, this implies that $\text{wd}_R(P) \leq n - 1$, which contradicts Lemma 14. Therefore $\alpha = 0$ and A is simple.

We now turn to considering factors of $R[\theta_1, \dots, \theta_u]/MR[\theta_1, \dots, \theta_u]$,

where M is a maximal ideal of R , and R is an algebra over the rationals. For conciseness, we here use the term *u-differential Ritt algebra* to stand for a commutative *u*-differential ring which is an algebra over the rationals. In such a case, the rings $R[\theta_1, \dots, \theta_j]$ will also be algebras over the rationals, but we do not refer to them as Ritt algebras since they are usually not commutative.

LEMMA 17. *Let R be any differential ring which is an algebra over the rationals, and let M be any maximal right ideal of R . If $(\delta + a)(M) \not\subseteq M$ for all $a \in R$, then $MR[\theta]$ is a maximal right ideal of $R[\theta]$.*

PROOF. Suppose on the contrary that $R[\theta]$ has a right ideal J such that $MR[\theta] < J < R[\theta]$, and pick an element $x \in J - MR[\theta]$ of minimal degree. Observing that $J \cap R = M$, we see that x must have degree $n > 0$, and we write $x = x_0 + \dots + x_n \theta^n$ with $x_0, \dots, x_n \in R$ and $x_n \neq 0$. In view of the minimality of n , we infer that $x_n \notin M$, whence $x_n r + y = 1$ for some $r \in R$, $y \in M$. Then $xr + y\theta^n$ has leading term θ^n , hence $xr + y\theta^n$ is an element of $J - MR[\theta]$ with degree n . Thus, replacing x by $xr + y\theta^n$, we may assume that $x_n = 1$.

Given any $m \in M$, it is clear that $xm - m\theta^n \in J$. Observing that $xm - m\theta^n$ has degree at most $n - 1$, we obtain $xm - m\theta^n \in MR[\theta]$, by the minimality of n . Since the coefficient of θ^{n-1} in $xm - m\theta^n$ is $x_{n-1}m + n(\delta m)$, we thus get $x_{n-1}m + n(\delta m) \in M$. But now $(\delta + x_{n-1}/n)(M) \subseteq M$, which is impossible.

LEMMA 18. *Let R be a u-differential Ritt algebra, and let M be a maximal ideal of R . Assume that s is a nonnegative integer such that the induced maps $\delta_1^*, \dots, \delta_s^* \in \text{Hom}_{R/M}(M/M^2, R/M)$ are linearly independent over R/M . Then $MR[\theta_1, \dots, \theta_s]$ is a maximal right ideal of $R[\theta_1, \dots, \theta_s]$.*

PROOF. We first prove the following series of statements $P_0, \dots, P_{s-1}$. P_j : If $a \in R[\theta_1, \dots, \theta_j]$ and $r_{j+1}, \dots, r_s \in R$ such that

$$(a + r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(M) \subseteq MR[\theta_1, \dots, \theta_j],$$

then $a \in R + MR[\theta_1, \dots, \theta_j]$ and $r_{j+1}, \dots, r_s \in M$.

To prove P_0 , assume that we have $a, r_1, \dots, r_s \in R$ such that $(a + r_1\delta_1 + \dots + r_s\delta_s)(M) \subseteq M$. Since $aM \subseteq M$ as well, we obtain $(r_1\delta_1 + \dots + r_s\delta_s)(M) \subseteq M$, for which it follows that $r_1\delta_1^* + \dots + r_s\delta_s^* = 0$. In view of the linear independence of $\delta_1^*, \dots, \delta_s^*$ over R/M , this implies that $r_1, \dots, r_s \in M$. Therefore P_0 holds.

Now let $0 < j \leq s - 1$ and assume that P_{j-1} holds. If P_j fails, then there exist elements $a \in R[\theta_1, \dots, \theta_j]$ and $r_{j+1}, \dots, r_s \in R$ such that

$$(a + r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(M) \subseteq MR[\theta_1, \dots, \theta_j],$$

but either $a \notin R + MR[\theta_1, \dots, \theta_j]$ or else some $r_i \notin M$. In case $a \in R + MR[\theta_1, \dots, \theta_j]$, then $aM \subseteq MR[\theta_1, \dots, \theta_j]$ and hence $(r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(M) \subseteq MR[\theta_1, \dots, \theta_j]$, from which we obtain $(r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(M) \subseteq M$. In this situation, however, P_0 says that $r_{j+1}, \dots, r_s \in M$, which is impossible. Thus we must have $a \notin R + MR[\theta_1, \dots, \theta_j]$, and in particular $a \neq 0$. We may also assume that a has the lowest degree in θ_j of those elements of $R[\theta_1, \dots, \theta_j]$ for which there exist $r_{j+1}, \dots, r_s \in R$ with

$$(a + r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(M) \subseteq MR[\theta_1, \dots, \theta_j].$$

Now write $a = a_0 + a_1\theta_j + \dots + a_k\theta_j^k$, where $a_0, \dots, a_k \in R[\theta_1, \dots, \theta_{j-1}]$ and $a_k \neq 0$. In view of P_{j-1} , we must have $k > 0$, and then it follows from the minimality of k that $a_k \notin MR[\theta_1, \dots, \theta_{j-1}]$.

If $k \geq 2$, then for any $m \in M$ we compute that

$$(a + r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(m)$$

leads off with the terms $a_k m \theta_j^k + [a_{k-1}m + ka_k(\delta_j m)]\theta_j^{k-1}$, from which we obtain

$$a_k m, a_{k-1}m + ka_k(\delta_j m) \in MR[\theta_1, \dots, \theta_{j-1}].$$

First, we have $a_k M \subseteq MR[\theta_1, \dots, \theta_{j-1}]$, hence P_{j-1} says that $a_k = r + b$ for some $r \in R$, $b \in MR[\theta_1, \dots, \theta_{j-1}]$. Inasmuch as $a_k \notin MR[\theta_1, \dots, \theta_{j-1}]$, we see that $r \notin M$. Second, we have $(a_{k-1} + ka_k\delta_j)(M) \subseteq MR[\theta_1, \dots, \theta_{j-1}]$, and clearly $(kb\delta_j)(M) \subseteq MR[\theta_1, \dots, \theta_{j-1}]$ as well, whence

$$(a_{k-1} + kr\delta_j)(M) \subseteq MR[\theta_1, \dots, \theta_{j-1}].$$

According to P_{j-1} , we obtain $kr \in M$, and then $r \in M$ (because R is a Ritt algebra). This is a contradiction.

Therefore $k < 2$, so the only possibility left is $k = 1$. Now $a = a_0 + a_1\theta_j$, hence for any $m \in M$ we have

$$\begin{aligned} & (a + r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(m) \\ &= a_1 m \theta_j + [a_0 m + a_1(\delta_j m) + r_{j+1}(\delta_{j+1} m) + \dots + r_s(\delta_s m)]. \end{aligned}$$

Thus $a_1 M \subseteq MR[\theta_1, \dots, \theta_{j-1}]$ and also

$$(a_0 + a_1\delta_j + r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(M) \subseteq MR[\theta_1, \dots, \theta_{j-1}].$$

As above, it follows from the first inclusion that $a_1 = r + b$ for some $r \in R - M$, $b \in MR[\theta_1, \dots, \theta_{j-1}]$, and then we infer from the second inclusion that

$$(a_0 + r\delta_j + r_{j+1}\delta_{j+1} + \dots + r_s\delta_s)(M) \subseteq MR[\theta_1, \dots, \theta_{j-1}].$$

But now P_{j-1} gives us $r \in M$, which is impossible.

Therefore P_j must hold, and the induction works. We now return to the proof of the lemma and show that for $j = 0, \dots, s$, $MR[\theta_1, \dots, \theta_j]$ is a maximal right ideal of $R[\theta_1, \dots, \theta_j]$. For $j = 0$, this is part of our hypotheses. Now let $0 < j \leq s$ and assume that $MR[\theta_1, \dots, \theta_{j-1}]$ is a maximal right ideal of $R[\theta_1, \dots, \theta_{j-1}]$. In view of P_{j-1} , we must have

$$(\delta_j + a)(MR[\theta_1, \dots, \theta_{j-1}]) \not\subseteq MR[\theta_1, \dots, \theta_{j-1}]$$

for all $a \in R[\theta_1, \dots, \theta_{j-1}]$, whence Lemma 17 shows that $MR[\theta_1, \dots, \theta_j]$ is a maximal right ideal of $R[\theta_1, \dots, \theta_j]$.

PROPOSITION 19. *Let R be a noetherian u -differential Ritt algebra, and set $T = R[\theta_1, \dots, \theta_u]$. Let M be any maximal ideal of R . If J is any right ideal of T which contains M , then $\text{pd}_T(T/J) \leq \text{rank}(M) + \text{diff dim}(M)$.*

PROOF. If $s = \text{diff codim}(M)$, then according to Proposition 12 the subspace W of $\text{Hom}_{R/M}(M/M^2, R/M)$ spanned by $\delta_1^*, \dots, \delta_u^*$ has dimension s ; hence we may arrange the indices $1, \dots, u$ so that $\delta_1^*, \dots, \delta_s^*$ is a basis for W . Setting $Q = R[\theta_1, \dots, \theta_s]$, we now see from Lemma 18 that MQ is a maximal right ideal of Q .

Given any $j \in \{s+1, \dots, u\}$, we must have $\delta_j^* = r_{j1}\delta_1^* + \dots + r_{js}\delta_s^*$ for suitable $r_{ji} \in R$, whence $(\delta_j - r_{j1}\delta_1 - \dots - r_{js}\delta_s)(M) \subseteq M$. Setting $q_j = r_{j1}\theta_1 + \dots + r_{js}\theta_s \in Q$, we compute that $(\theta_j - q_j)M \subseteq MT$. If now X denotes the set of all products of nonnegative powers of $\theta_{s+1} - q_{s+1}, \dots, \theta_u - q_u$, then we obtain $XMT \subseteq MT$.

In particular, $XM_Q \subseteq MT \subseteq J$. Observing that T is generated as a right Q -module by X , we infer that $(T/J)_Q$ is a sum of homomorphic images of Q/MQ . Inasmuch as Q/MQ is a simple right Q -module, it follows that $(T/J)_Q$ is isomorphic to a direct sum of copies of Q/MQ , whence $\text{pd}_Q(T/J) \leq \text{pd}_Q(Q/MQ)$. Since $Q/MQ \cong (R/M) \otimes_R Q$, we also have $\text{pd}_Q(Q/MQ) \leq \text{pd}_R(R/M)$. In addition, $\text{pd}_R(R/M) = \text{rank}(M)$ by Proposition 9, and thus $\text{pd}_Q(T/J) \leq \text{rank}(M)$. According to Proposition 1, $\text{pd}_T(T/J) \leq u - s + \text{rank}(M)$. Inasmuch as $u - s = \text{diff dim}(M)$, this gives us the required inequality.

5. Global dimension formulas.

THEOREM 20. *Let R be a noetherian u -differential Ritt algebra with $\text{gl dim } R < \infty$. Then*

$$\begin{aligned} r \text{ gl dim } R[\theta_1, \dots, \theta_u] \\ = \sup\{\text{rank}(P) + \text{diff dim}(P) \mid P \text{ is a prime ideal of } R\}. \end{aligned}$$

PROOF. If $S = R[\theta_1, \dots, \theta_u]$, $n = r \text{ gl dim } S$, and

$$k = \sup\{\text{rank}(P) + \text{diff dim}(P) \mid P \text{ is a prime ideal of } R\},$$

then $n \geq k$ by Proposition 15. According to Proposition 2, $n \leq u + \text{gl dim } R < \infty$. Inasmuch as S is right and left noetherian, Theorem 16 says that there exists a simple right S -module A with $\text{pd}_S(A) = n$, and we note that $\text{wd}_S(A) = n$ also.

Choose a nonzero element $x \in A$ whose R -annihilator $P = \{r \in R \mid xr = 0\}$ is maximal among the R -annihilators of all nonzero elements of A . According to [10, Theorem 6], P is a prime ideal of R . If $T = R_P[\theta_1, \dots, \theta_u]$, then the right R_P -module A_P can be made into a right T -module by defining $(a/s)\theta_i = [a\theta_i s + a(\delta_i s)]/s^2$ for all i and all $a/s \in A_P$. Since the R -annihilator of x is P , the natural map $A \rightarrow A_P$ is not zero. However, this map is an S -homomorphism and A is a simple S -module, hence $A \rightarrow A_P$ must be a monomorphism. In view of Lemma 14, we thus obtain $\text{wd}_S(A_P) = n$.

Now $A = xS$ and thus $A_P = (x/1)T$, from which we infer that $A_P \cong T/J$ for some right ideal J of T which contains PR_P . According to Proposition 19, $\text{pd}_T(A_P) \leq \text{rank}(PR_P) + \text{diff dim}(PR_P)$. In view of Corollary 13, we now obtain $\text{wd}_T(A_P) \leq \text{rank}(P) + \text{diff dim}(P) \leq k$. Inasmuch as T_S is flat by Proposition 3, $\text{wd}_S(A_P) \leq \text{wd}_T(A_P)$, and therefore $n \leq k$.

THEOREM 21. *Let R be any commutative noetherian u -differential ring such that $\text{gl dim } R < \infty$. Set*

$$\begin{aligned} k &= \sup\{\text{rank}(P) + \text{diff dim}(P) \mid P \text{ is a prime ideal of } R \text{ and } \text{char}(R/P) = 0\}, \\ q &= \sup\{\text{rank}(M) \mid M \text{ is a maximal ideal of } R \text{ and } \text{char}(R/M) > 0\}. \end{aligned}$$

[In either case, if there are no ideals of the type required, the supremum is considered to be $-\infty$.] Then

$$r \text{ gl dim } R[\theta_1, \dots, \theta_u] = \max\{k, q + u\}.$$

PROOF. In view of Propositions 10 and 15, we have $r \text{ gl dim } R[\theta_1, \dots, \theta_u] \geq \max\{k, q + u\}$. According to Proposition 4, the reverse inequality will hold

provided $\text{r gl dim } R_M[\theta_1, \dots, \theta_u] \leq \max\{n, q + u\}$ for each maximal ideal M of R .

First consider the case when $\text{char}(R/M) > 0$. According to Proposition 9, $\text{gl dim } R_M = \text{rank}(M) \leq q$, hence Proposition 2 shows that $\text{r gl dim } R_M[\theta_1, \dots, \theta_u] \leq q + u$.

Now assume that $\text{char}(R/M) = 0$. Here $nR_M \not\subseteq MR_M$ for all nonzero integers n , hence all nonzero integers are invertible in R_M . Thus R_M is a Ritt algebra, and so Theorem 20 is applicable. According to [10, Theorem 34], any prime ideal of R_M must have the form PR_M for some prime ideal P of R which is contained in M , and since $\text{char}(R/M) = 0$ we see that $\text{char}(R/P) = 0$, too. In view of Corollary 13, we obtain

$$\text{rank}(PR_M) + \text{diff dim}(PR_M) = \text{rank}(P) + \text{diff dim}(P) \leq k,$$

and therefore Theorem 20 shows that $\text{r gl dim } R_M[\theta_1, \dots, \theta_u] \leq k$. $\square$

In particular, Theorem 21 gives a formula for the global dimension of $R[\theta]$ when R is only a 1-differential ring. For this case, the formula can be improved somewhat as follows, since the differential dimension of any prime ideal P depends only on whether or not P is a differential ideal. Also, for this case it is possible to restrict attention to just the maximal ideals of R .

THEOREM 22. *Let R be any commutative noetherian differential ring with $\text{gl dim } R = n < \infty$. Let $\mathcal{M}$ denote the collection of all differential maximal ideals of R , together with all maximal ideals M such that $\text{char}(R/M) > 0$, and set $k = \sup\{\text{rank}(M) \mid M \in \mathcal{M}\}$. [If $\mathcal{M}$ is empty, then k is considered to be $-\infty$.] Then $\text{r gl dim } R[\theta] = \max\{n, k + 1\}$.*

PROOF. According to Proposition 2, $\text{r gl dim } R[\theta] \geq n$. Inasmuch as $\text{diff dim}(M) = 1$ for any differential maximal ideal M of R , Theorem 21 shows that $\text{r gl dim } R[\theta] \geq k + 1$.

Suppose that P is any prime ideal of R with $\text{char}(R/P) = 0$. If P is not maximal, then it is clear from Proposition 9 that $\text{rank}(P) < n$. Since $\text{diff dim}(P) \leq 1$, we get $\text{rank}(P) + \text{diff dim}(P) \leq n$ in this case. Now assume that P is a maximal ideal. If P is not a differential ideal, then $\text{diff dim}(P) = 0$ and $\text{rank}(P) + \text{diff dim}(P) \leq n$, using Proposition 9 again. On the other hand, if P is a differential ideal, then $\text{rank}(P) + \text{diff dim}(P) = 1 + \text{rank}(P) \leq k + 1$, by definition of k .

Thus we have $\text{rank}(P) + \text{diff dim}(P) \leq \max\{n, k + 1\}$ for all prime ideals P of R such that $\text{char}(R/P) = 0$. In view of Theorem 21, we conclude that $\text{r gl dim } R[\theta] \leq \max\{n, k + 1\}$.

We conclude this section by using Theorem 22 to derive a formula for the global dimension of $R[\theta]$ which involves only differential ideals of R . We recall

that a proper ideal J in a commutative ring R is said to be *primary* provided all zero-divisors in the ring R/J are nilpotent.

THEOREM 23. *Let R be any commutative noetherian differential ring with $\text{gl dim } R = n < \infty$, and set $k = \sup\{\text{pd}_R(R/J) \mid J \text{ is a primary differential ideal of } R\}$. [If R has no primary differential ideals, then k is considered to be $-\infty$.] Then $\text{r gl dim } R[\theta] = \max\{n, k + 1\}$.*

PROOF. According to Proposition 2, $\text{r gl dim } R[\theta] \geq n$. Inasmuch as R is semiprime by Proposition 9, Corollary 8 shows that $\text{r gl dim } R[\theta] \geq k + 1$.

Now consider any maximal ideal M of R such that $\text{char}(R/M) = p > 0$. If J is the ideal of R generated by pR and $\{x^p \mid x \in M\}$, then as in Proposition 10 we see that M/J is nilpotent and that $\text{pd}_R(R/J) = \text{pd}_R(R/M)$. Inasmuch as M/J is nilpotent, R/J must be local, from which we infer that J is a primary ideal of R . Also, J is clearly a differential ideal, whence $\text{pd}_R(R/J) \leq k$. Since $\text{pd}_R(R/M) = \text{rank}(M)$ by Proposition 9, we thus obtain $\text{rank}(M) \leq k$.

Thus we have $\text{rank}(M) \leq k$ for all maximal ideals M of R such that $\text{char}(R/M) > 0$. Since any differential maximal ideal M of R is a primary differential ideal, we also have $\text{rank}(M) \leq k$ for all differential maximal ideals M . According to Theorem 22, we thus obtain $\text{r gl dim } R[\theta] \leq \max\{n, k + 1\}$.

6. Applications. For any ring S and any positive integer u , the *Weyl algebra* of degree u over S is the ring $A_u(S) = S[x_1, \dots, x_u][\theta_1, \dots, \theta_u]$, where the x_i are ordinary polynomial indeterminates, and we use the derivations $\delta_i = \partial/\partial x_i$ on $S[x_1, \dots, x_u]$. J.-E. Roos has shown that for a field F of characteristic 0, $\text{r gl dim } A_u(F) = u$ [13, Théorème 1], while G. S. Rinehart has shown that, for a field F of positive characteristic, $\text{r gl dim } A_u(F) = 2u$ [11, Theorem, p. 345]. We generalize these results in the following theorem, which has also been proved (using entirely different methods) in [12, Theorem 2.6].

THEOREM 24. *Let S be any commutative noetherian ring with $\text{gl dim } S = n < \infty$, and set $k = \sup\{\text{rank}(M) \mid M \text{ is a maximal ideal of } S \text{ and } \text{char}(S/M) > 0\}$. [If S has no such maximal ideals, then k is considered to be $-\infty$.] Then for any positive integer u , $\text{r gl dim } A_u(S) = \max\{n + u, k + 2u\}$.*

PROOF. Set $R = S[x_1, \dots, x_u]$ and $\delta_i = \partial/\partial x_i$ for $i = 1, \dots, u$. Since $\text{gl dim } R = n + u$, Proposition 2 shows that $\text{r gl dim } A_u(S) \geq n + u$.

If S has any maximal ideals M such that $\text{char}(S/M) > 0$, then we may choose such an M with $\text{rank}(M) = k$. Inasmuch as S/M is a field, the ring $R/MR \cong (S/M)[x_1, \dots, x_u]$ has Krull dimension u , whence R/MR must have a maximal ideal K/MR of rank u . Then K is a maximal ideal of R such that $\text{char}(R/K) > 0$, and clearly $\text{rank}(K) \geq k + u$, hence Theorem 21 says that $\text{r gl dim } A_u(S) \geq k + 2u$.

Therefore $\text{r gl dim } A_u(S) \geq \max\{n + u, k + 2u\}$. According to Theorem 21, to prove the reverse inequality it is enough to show that $\text{rank}(M) \leq k + u$ for any maximal ideal M of R with $\text{char}(R/M) > 0$, and that $\text{rank}(P) + \text{diff dim}(P) \leq n + u$ for any prime ideal P of R such that $\text{char}(R/P) = 0$.

First consider any maximal ideal M of R for which $\text{char}(R/M) > 0$. Choosing a maximal ideal K of S which contains $S \cap M$, we have $\text{char}(S/K) > 0$ and so $\text{rank}(S \cap M) \leq \text{rank}(K) \leq k$. By induction on [10, Theorem 149], we find that $\text{rank}(M) \leq k + u$.

Now consider any prime ideal P of R with $\text{char}(R/P) = 0$, and set $s = \text{diff dim}(P)$. If $T = R_P$, $M = PR_P$, and W is the subspace of $\text{Hom}_{T/M}(M/M^2, T/M)$ spanned by $\delta_1^*, \dots, \delta_u^*$, then by Proposition 12 W has dimension $u - s$. Thus we may arrange the indices $1, \dots, u$ so that $\delta_{s+1}^*, \dots, \delta_u^*$ is a basis for W . Set $Q = P \cap (S[x_1, \dots, x_s])$ and note that $S[x_1, \dots, x_s]/Q$ has characteristic 0. We claim that $\delta_i(Q) \subseteq Q$ for $i = 1, \dots, s$. Given $1 \leq i \leq s$, we must have $\delta_i^* = t_{s+1}\delta_{s+1}^* + \dots + t_u\delta_u^*$ for suitable $t_j \in T$. Multiplying out the denominators in this equation, we obtain

$$a\delta_i^* = r_{s+1}\delta_{s+1}^* + \dots + r_u\delta_u^* \quad \text{for some } a \in R - P, r_{s+1}, \dots, r_u \in R.$$

Thus $(a\delta_i - r_{s+1}\delta_{s+1} - \dots - r_u\delta_u)(M) \subseteq M$, from which we infer that

$$(a\delta_i - r_{s+1}\delta_{s+1} - \dots - r_u\delta_u)(P) \subseteq P.$$

Since $Q \subseteq P$ and $\delta_{s+1}, \dots, \delta_u$ all vanish on Q , we thus obtain $a\delta_i(Q) \subseteq P$. Now P is a prime ideal of R and $a \in R - P$, hence it follows that $\delta_i(Q) \subseteq P$, from which we conclude that $\delta_i(Q) \subseteq Q$, as claimed.

All of the rings $S[x_1, \dots, x_i]/(Q \cap S[x_1, \dots, x_i])$ ($i = 1, \dots, s$) have characteristic 0, hence with the help of the relations $\delta_i(Q) \subseteq Q$ an easy induction shows that $Q \cap (S[x_1, \dots, x_i]) = (Q \cap S)[x_1, \dots, x_i]$ for each $i = 1, \dots, s$. Consequently $Q = (Q \cap S)[x_1, \dots, x_s]$, whence [10, Theorem 149] shows that $\text{rank}(Q) = \text{rank}(Q \cap S)$. That same theorem also shows that $\text{rank}(P) \leq u - s + \text{rank}(Q)$, and it is clear from Proposition 12 that $\text{rank}(Q \cap S) \leq n$, hence we obtain $\text{rank}(P) \leq n + u - s$. Therefore $\text{rank}(P) + \text{diff dim}(P) \leq n + u$.

COROLLARY 25. *Let S be any commutative noetherian ring with $\text{gl dim } S = n < \infty$, and let u be any positive integer. If S is an algebra over the rationals, then $\text{r gl dim } A_u(S) = n + u$.*

Corollary 25 has also been obtained in [1, Corollary 2.6].

Given any ring S and any positive integer u , then following [2] we can define a ring $F_u(S) = S[[x_1, \dots, x_u]] [\theta_1, \dots, \theta_u]$ analogous to the Weyl algebra $A_u(S)$. If S is a commutative noetherian ring with $\text{gl dim } S = n < \infty$, and if S is an algebra over the rationals, then J.-E. Björk has shown in [2, The-

orem 4.2] that $\text{r gl dim } F_u(S) = n + u$. We shall generalize this result, but first some facts about power series rings must be developed. [We note that our proofs do not depend on Björk's result, and our methods are completely different from his.]

LEMMA 26. *Let S be a commutative noetherian ring with $\text{gl dim } S = n < \infty$. If u is any positive integer, then $\text{gl dim } S[[x_1, \dots, x_u]] = n + u$.*

PROOF. It obviously suffices to prove the case $n = 1$. The indeterminate x lies in the Jacobson radical of $S[[x]]$, and x is not a zero-divisor in $S[[x]]$. Since $S[[x]]/xS[[x]] \cong S$, [9, Part III, Theorem 10] shows that $\text{r gl dim } S[[x]] = n + 1$.

LEMMA 27. *If F is a field and u any positive integer, then $F[[x_1, \dots, x_u]]$ has Krull dimension u .*

PROOF. This is immediate from Lemma 26 and Proposition 9. $\square$

For use in the next lemma, we recall that if S is a commutative local ring with maximal ideal M , then $S[[x]]$ is a local ring with maximal ideal generated by M and x . Clearly, the ideal J of $S[[x]]$ generated by M and x is a maximal ideal. Also, if p is any element of $S[[x]]$ which does not belong to J , then the constant term of p is not in M and so is invertible in S , whence p is invertible in $S[[x]]$.

LEMMA 28. *Let S be a commutative noetherian ring, and let Q be any prime ideal of S , u any positive integer. Then $Q[[x_1, \dots, x_u]]$ is a prime ideal of $S[[x_1, \dots, x_u]]$ with rank equal to $\text{rank}(Q)$. Also, if P is any prime ideal of $S[[x_1, \dots, x_u]]$ such that $P \cap S = Q$, then $\text{rank}(P) \leq u + \text{rank}(Q)$.*

PROOF. We may obviously assume that $\text{rank}(Q) < \infty$. Also, we clearly need only prove the case $u = 1$. Finally, since P is disjoint from $S - Q$, all the ranks we are interested in remain the same after localizing at Q , hence we may assume, without loss of generality, that S is local with maximal ideal Q . As remarked above, it follows that $S[[x]]$ is local with maximal ideal M generated by Q and x .

Now M is a prime ideal in the noetherian ring $S[[x]]$, and x is an element of M which is not a zero-divisor in $S[[x]]$, hence [10, Theorem 155] says that the rank of $M/xS[[x]]$ in $S[[x]]/xS[[x]]$ equals $\text{rank}(M) - 1$. Inasmuch as $S[[x]]/xS[[x]] \cong S$, we infer that $\text{rank}(M/xS[[x]]) = \text{rank}(Q)$, and thus $\text{rank}(M) = 1 + \text{rank}(Q)$. Observing that $P \subseteq M$, we obtain $\text{rank}(P) \leq 1 + \text{rank}(Q)$. Finally, since $\text{rank}(M) = 1 + \text{rank}(Q) < \infty$ and $Q[[x]]$ is properly contained in M , we must have $\text{rank}(Q[[x]]) \leq \text{rank}(Q)$, from which we conclude that $\text{rank}(Q[[x]]) = \text{rank}(Q)$. $\square$

With the help of these three lemmas, we may use the proof of Theorem 24, mutatis mutandis, to prove the following generalization of Björk's theorem:

THEOREM 29. *Let S be any commutative noetherian ring with $\text{gl dim } S = n < \infty$, and set $k = \sup\{\text{rank}(M) \mid M \text{ is a maximal ideal of } S \text{ and } \text{char}(S/M) > 0\}$. [If S has no such maximal ideals, then k is considered to be $-\infty$.] Then, for any positive integer u ,*

$$\text{r gl dim } F_u(S) = \max\{n + u, k + 2u\}.$$

J. Cozzens and J. Johnson have shown that for any u -differential field F , $\text{r gl dim } F[\theta_1, \dots, \theta_u] = u$ [4, Theorem 1(b)]. In view of Corollary 8 and Proposition 2, this result generalizes to semisimple artinian rings:

THEOREM 30. *If R is any semisimple artinian u -differential ring, then $\text{r gl dim } R[\theta_1, \dots, \theta_u] = u$.*

REFERENCES

1. S. M. Bhatwadekar, *On the global dimension of Ore-extensions*, Nagoya Math. J. **50** (1973), 217–225. MR **47** #8515.
2. J.-E. Björk, *The global homological dimension of some algebras of differential operators*, Invent. Math. **17** (1972), 67–78. MR **47** #8619.
3. S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473. MR **22** #11017.
4. J. Cozzens and J. Johnson, *Some applications of differential algebra to ring theory*, Proc. Amer. Math. Soc. **31** (1972), 354–356. MR **44** #6662.
5. K. R. Goodearl, *Global dimension of differential operator rings*, Proc. Amer. Math. Soc. **45** (1974), 315–322.
6. N. S. Gopalakrishnan and R. Sridharan, *Homological dimension of Ore-extensions*, Pacific J. Math. **19** (1966), 67–75. MR **34** #223.
7. R. Gordon and J. C. Robson, *Krull dimension*, Mem. Amer. Math. Soc. No. 133 (1973).
8. I. N. Herstein, *Noncommutative rings*, Carus Math. Monographs, no. 15, Math. Assoc. Amer.; distributed by Wiley, New York, 1968. MR **37** #2790.
9. I. Kaplansky, *Fields and rings*, Univ. of Chicago Press, Chicago, Ill., 1969. MR **42** #4345.
10. ———, *Commutative rings*, Allyn and Bacon, Boston, Mass., 1970. MR **40** #7234.
11. G. S. Rinehart, *Note on the global dimension of a certain ring*, Proc. Amer. Math. Soc. **13** (1962), 341–346. MR **25** #1196.
12. G. S. Rinehart and A. Rosenberg, *The global dimension of Ore extensions and Weyl algebras*, A Collection of Papers in Honor of Samuel Eilenberg, Academic Press, New York, 1975.
13. J.-E. Roos, *Détermination de la dimension homologique globale des algèbres de Weyl*, C. R. Acad. Sci. Paris Sér. A–B **274** (1972), A23–A26. MR **45** #1995.
14. F. L. Sandomierski, *Semisimple maximal quotient rings*, Trans. Amer. Math. Soc. **128** (1967), 112–120. MR **35** #5473.